could then be added to the controller, as was done by Gautam and Mutharasan.

The two integral-equivalent modes, while different in design, serve the same function. But the use of an adjustable integral coefficient might provide more flexibility for the Gautam and Mutharasan approach. It is difficult to assess the relative merits of the two integral approaches from the brief simulation study described here.

The features noted above suggest several new model-based controller algorithms for time-delay processes. In addition, there are other reported model-based algorithms that should be considered. These include the discrete version of the classical Smith predictor (Meyer et al.), various prototype algorithms (Chiu et al. 1973), and the algorithms of Takahashi and colleagues (Tomizuka et al. 1978, Auslander et al. 1978). The last group of algorithms is of interest because of the development of a simplified z-domain process model based on step response data.

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Fluid Pressure Distribution in an Aerated Hopper

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Aeration of hoppers is now widely used, both to increase the discharge rate of solids and as a flow promoter for non-free flowing materials. Any theoretical approach to the stress analysis or to the prediction of discharge rates from an aerated hopper needs the fluid pressure distribution in the hopper. In this note, an analytical expression for the fluid pressure distribution in an aerated hopper is developed.

THEORY

It is assumed that before gas is added to the hopper, there is zero fluid pressure gradient in the particle bed. That is, the discharge rate at gravity flow is not affected by the presence of the interstitial gas. This assumption is quite reasonable, for particles greater than about 100 μm in diameter. It should, however, be borne in mind that for fine particles, the negative gauge pressure created during the flow is appreciable, and it affects the discharge rate of solids. Thus, we conclude that the relative velocity u between the fluid and the particles is caused by the additional gas input through the walls (Papazoglou and Pyle 1970).

The gas is introduced into the hopper through a porous conical aeration section, shown in Figure 1. Gas velocity entering the bed from this conical distributor is assumed to be uniform. According to Darcy's Law, the fluid pressure gradient in a particle bed is proportional to the relative velocity between the gas and the particles (Davidson and Harrison 1963). Therefore the components of the interstitial gas velocity are given by

$$u_r = v_r - K_p \frac{\partial p}{\partial r} \tag{1}$$

$$u_\theta = v_\theta - \frac{K_p}{r} \frac{\partial p}{\partial \theta} \tag{2}$$

With the bulk density of the bed assumed constant (Altiner 1975), the continuity equations for the particles and the gas are (Bird et al. 1960)

for particles

$$\nabla \cdot v = \frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} = 0 \tag{3}$$

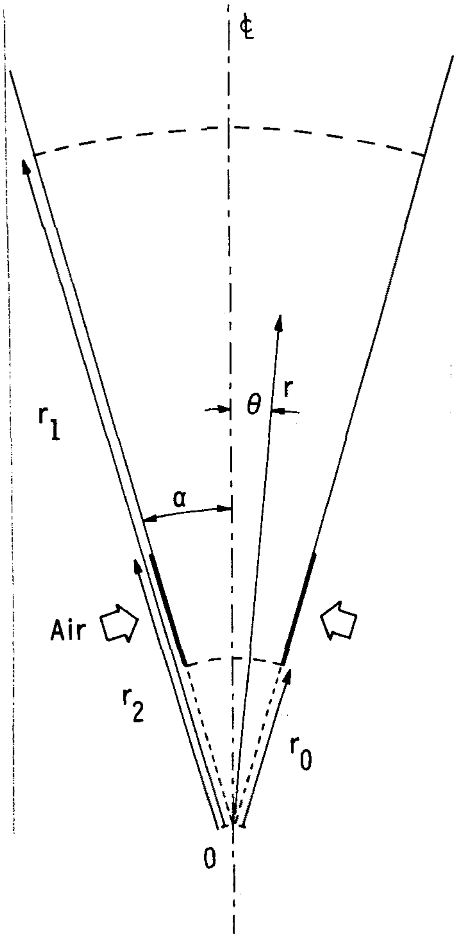

Figure 1. Hopper configuration.

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for gas

$$\nabla \cdot u = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta \cot\theta}{r} = 0 \quad (4)$$

From Equations (1-4) a differential equation for the fluid pressure is obtained.

$$\nabla^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial p}{\partial \theta} = 0 \quad (5)$$

This is, of course, the Laplace Equation with axial symmetry. The boundary conditions are

$$r = r_0 \qquad p = 0 \qquad \text{for all } \theta$$
$$r = r_1 \qquad p = 0 \qquad \text{for all } \theta$$
$$\theta = 0 \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \text{for all } r$$

at $\theta = \alpha$ the boundary consists of two parts: For $r > r_2$, where no gas is being injected into the hopper, there is no component of velocity normal to the walls, therefore $(\partial p/\partial\theta)_\alpha = 0$. For $r \leq r_2$, the region of air injection, there is a finite velocity component normal to the walls. Therefore, from Darcy's Law we can write

$$\frac{1}{r}\left(\frac{\partial p}{\partial \theta}\right)_\alpha = \frac{U_\alpha}{\epsilon K_p} = \text{constant}$$

The solution of Equation (5) with the above boundary conditions is not found in any textbook, and it is worthwhile to present the major steps. Assuming the pressure of the form $p = f(r) \cdot g(\theta)$ and separating the variables gives

$$\frac{1}{f}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) = -\frac{1}{g \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta}\right) = -\frac{k^2+1}{4} \quad (6)$$

Solving for $f(r)$ is straightforward; with transformations $r/r_0 = x$ and $x = e^s$

$$f(s) = e^{-s/2}\left(A_1 \cos\frac{ks}{2} + B_1 \sin\frac{ks}{2}\right) \quad (7)$$

k, A_1 and B_1 are constants to be determined. Writing the equation for $g(\theta)$ as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \lambda(\lambda+1)g = 0 \quad (8)$$

where λ is related to k by $\lambda(\lambda + 1) = -(k^2 + 1)/4$. With transformation $\omega = \cos\theta$ Equation (8) becomes

$$(1 - \omega^2)\frac{d^2 g}{d\omega^2} - 2\omega\frac{dg}{d\omega} + \lambda(\lambda+1)g = 0 \quad (9)$$

There exists exactly one solution of Equation (9) which is finite at $\omega = 1$ i.e. $\theta = 0$, the centerline (Dixon 1974). It is denoted by $P_\lambda(\omega)$ the Legendre function and is defined for all complex λ. The solution is also finite at $\theta = \pi$ only if λ is positive integer. But, we are not interested in this region, since the θ's we are dealing with are much smaller than $\pi/2$.

$P_\lambda(\omega)$ is given by

$$P_\lambda(\cos\theta) = F_\lambda\left(-\lambda,\ \lambda+1;\ 1;\ \sin^2\frac{\theta}{2}\right) \quad (10)$$

where

$$F(a, b; c; z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} z^3 + \ldots$$

$$a = -\lambda, \quad b = \lambda + 1, \quad c = 1, \quad z = \sin^2\frac{\theta}{2}$$

Thus

$$P_\lambda(\cos\theta) = \sum_{m=1}^{\infty} \frac{(-1)^m \Lambda(\Lambda - 1.2.)(\Lambda - 2.3.)\ldots[(\Lambda - m(m-1))]}{(m!)^2} z^m \quad (11)$$

where

$$\Lambda = \lambda(\lambda + 1)$$

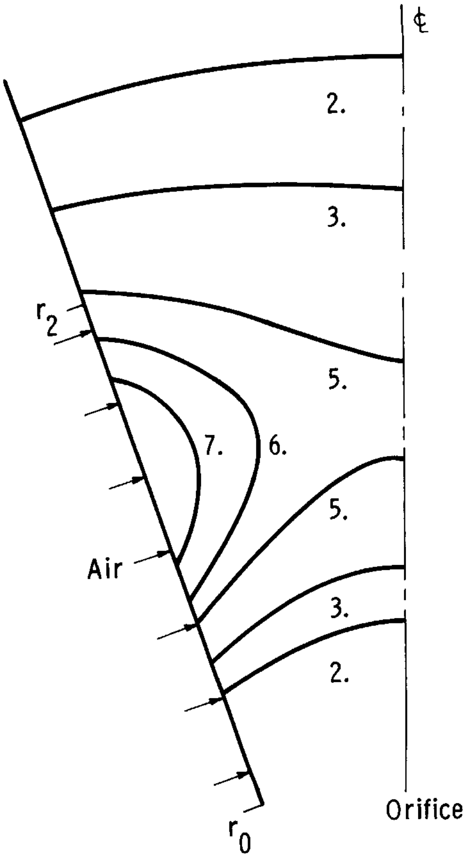

Figure 2. Fluid pressure contours near orifice.

Hence the general solution of the pressure equation is

$$p = e^{-s/2}\left(A_1 \cos\frac{ks}{2} + B_1 \sin\frac{ks}{2}\right) P_\lambda(\cos\theta) \quad (12)$$

Applying the boundary conditions gives (Altiner 1975)

$$p = \sum_{n=1}^{\infty} B_n e^{-s/2} \sin\frac{k_n s}{2} P_\lambda(\cos\theta) \quad (13)$$

where

$$k_n = 2\pi n/s_1 \quad n = 1, 2, 3 \ldots; \quad s = \ln x$$

$$B_n = \frac{4a_\alpha[P_\lambda'(\cos\alpha)]^{-1}}{s_1(k_n^2 + 9)}\left[e^{3s_2/2}\left(3 \sin\frac{k_n s_2}{2} - k_n \cos\frac{k_n s_2}{2}\right) + k_n\right]$$

$$a_\alpha = r_0 U_\alpha / K_p \epsilon$$

$$P_\lambda'(\cos\alpha) = \left.\frac{dP_\lambda(\cos\theta)}{d\theta}\right|_{\theta=\alpha}$$

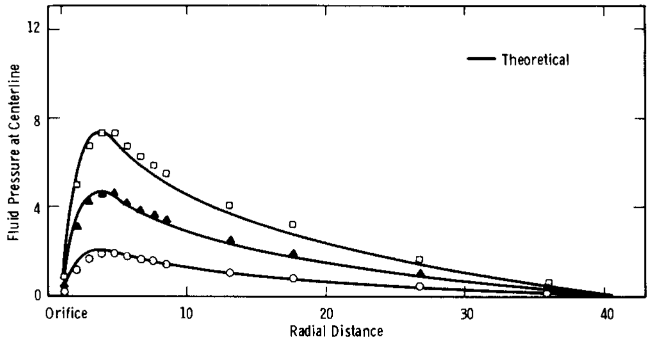

Figure 3. Experimental vs. theoretical fluid pressure.

DISCUSSION AND CONCLUSION

A typical pressure distribution predicted by the above equation is plotted in arbitrary units (Figure 2). In the vicinity of the aeration section, the change of pressure with θ is considerable. Moving up from the aeration section, however, the change with θ becomes smaller, as the aerating gas is more evenly distributed across the hopper. There are no experimental data available showing the variation of fluid pressure with both θ and r. However the experimental measurements by the author, of the fluid pressure along the centerline of a discharging plane hopper for various air inputs were compared with the corresponding equations for the two dimensional case.

The test hopper consisted of two inclined walls held together by two parallel plexiglass plates. It was arranged so that the hopper angle and the orifice width could be varied. Aeration was achieved by blowing air through two porous sections on the inclined walls extending from r_0 to r_2 (Figure 1). The results were obtained with sand of 276 μm surface-volume mean diameter and 2,640 kg/m^3 solid density. Permeability constant given by Carman-Kozeny expression, $K_p = d^2\phi^2\epsilon^2/180(1-\epsilon)^2\mu$ was used in the calculations (Carman 1937). Bed voidage of 0.522 was obtained by curve fitting one set of results, and was used for all the other air flow rates. Sphericity of sand was taken as 0.8.

In Figure 3, experimental results for three different aeration rates are shown in arbitrary units, together with the calculated pressure from the above theory. Agreement between the theoretical predictions and the experimental results is very good, and so supports the basic approach to the present theory. The close agreement with theory along the whole height of the hopper also verifies the assumption of constant hopper voidage. It is hoped that this analysis of the fluid pressure distribution will serve as a useful tool in the analytical study of aerated hoppers, whether the concern is the stress analysis, solids discharge rate, or any other aspect where fluid pressure has an influence.

ACKNOWLEDGMENT

Valuable discussions with Professor J. F. Davidson are gratefully acknowledged.

NOTATION

d = mean particle diameter
K_p = permeability constant
p = fluid pressure
r = radial coordinate
r_0, r_1 = radial positions of orifice and top surface respectively
r_2 = radial position of the end of aeration section
s = $\ln(r/r_0)$
u = interstitial fluid velocity
U_α = gas flowrate per unit area of aeration section
v = particle velocity
x = r/r_0

Greek Letters

α = half included hopper angle
ϵ = bed voidage
ϕ = sphericity of particles
θ = angular coordinate
λ = constant defined by Equation (8)
μ = fluid viscosity
ω = $\cos\theta$
Λ = $\lambda(\lambda + 1)$

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Effect of an Array of Objects on Mass Transfer Rates to the Tube Wall: An Additional Note

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In previous publications (Končar-Djurdjević and Duduković 1977, Duduković and Končar-Djurdjević 1979) we described the effect of a single object and of an array of disks on mass transfer rates to the wall of a coaxial cylindrical tube. In this note we intend to give a short review of experimental results obtained in the case of an array of spheres, as well as to indicate the differences of the effects of an array of disks and the array of spheres on mass transfer rates to the inside wall of a coaxial cylinder (Figure 1).

The apparatus used, measurement method, and all working conditions are the same as described (Duduković and Končar-Djurdjević 1979).

The experiments were carried out for a pair of spheres, each 47.6mm in diameter, i.e. for the ratio of sphere to tube diameter $d_p/d = 0.793$ and at the Reynolds number of $Re = 47{,}600$, for